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Workshop Documents

Middle School CCSS Math Workshop

Understanding the Math Standards

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1. Introduction

1.1. **What is Mathematics?** There is no agreed upon definition of Mathematics, but here is my philosophy...Mathematics is the language we use to describe the state of reality we exist in. Mathematics can be used to develop students into critical thinkers in a way that no other subject in school can.

1.2. **Guidelines to Understanding Standards.** While I can't give you a formula to help you understand each math standard, I can provide things to think about when analyzing a standard.

(1) What is the concept and the grammar around the concept?
(2) Why does this concept matter?
(3) What are the vocabulary words (i.e. terms) they need to know?
(4) What are the vocabulary words (i.e. terms) they need to learn?
(5) What technical skills are required to perform the calculations?
(6) What concepts must students have a deep understanding of to understand this concept (not just perform calculations)?
(7) What important properties, rules, or identities do students need to learn about the concept? When possible deduce these instead of presenting them.
(8) What is the next step (how will this concept be expanded on in future lessons and grades)?
(9) What are common misconceptions?

2. 8^{th} grade standard

8.EE.A.1 Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, $3^2 \times 3^{-5} = 3^{-3} = (\frac{1}{3})^3 = \frac{1}{27}$.

2.1. **What is the concept and the grammar around the concept?** This standard is about exponentiation. In particular how to take rational numbers to an integer power. Common expressions like 3^4 are read "three to the forth power" and 2^{-5} is "two to the negative fifth power".

2.2. **Why does this concept matter?**

- This is a step towards writing and understanding complex expressions. If you're interested in architecture, physics, biology, engineering, and many others.
- When in doubt...google it. http://passyworldofmathematics.com/exponents-in-the-real-world/

2.3. **What are the vocabulary words (i.e. terms) they need to know?**

- Integer (which may require reviewing positive and negative whole numbers)
- Fraction (which implies numerator and denominator)
- Rational number
- Reciprocal (multiplicative inverse)
- (Mathematical) Expression
- Numerical Expression
- Equivalent (Numerical Expressions)

Let's define these terms.

Definition 2.3.1. An **integer** is a number that can be written without a fractional component.

Notation 2.3.2. The symbol for the set of all integers is $\mathbb{Z}$.

Remark 2.3.3. There is no formal definition of a fraction that would work at this level. So students should be able to recognize a fraction, when something is not a fraction, identify the numerator and denominator.

Definition 2.3.4. A **rational number** is a number that can be written as $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

Notation 2.3.5. The symbol for the set of all rational numbers is $\mathbb{Q}$.

Remark 2.3.6. This would be a good time to make sure the students know the difference between a fraction and a rational number.

Definition 2.3.7. The **reciprocal (multiplicative inverse)** of a number x is a number, y, such that $x \times y = 1$.

Notation 2.3.8. The reciprocal of a number x is denoted $\frac{1}{x}$ or x^{-1}.

Remark 2.3.9. If the students are familiar with the notation x^{-1} you may want to get them to start wondering why it's denoted that way. This standard should answer this question for them.

Definition 2.3.10. A **(mathematical) expression** is a mathematical phrase.

Remark 2.3.11. The definition above may seem unsatisfying and not mathematical, but remember mathematics is a language. Being an expression is a syntactic concept. An expression must be well-formed: the operators must have the correct number of inputs in the correct places, the characters that make up these inputs must be valid, etc. Strings of symbols that violate the rules of syntax are not well-formed and are not valid mathematical expressions.

Example 2.3.12. Expressions: $2 + 3$, $\frac{x}{y}$, 7
Not Expressions: $\frac{}{4}$, $x-$, $3\times$

Definition 2.3.13. A **numerical expression** is an expression involving only numbers and operation symbols.

Example 2.3.14. Numerical Expressions: $2 + 3$, 7
Not Numerical Expression: $\frac{x}{y}$

Definition 2.3.15. Two numerical expressions are **equivalent** if they have the same value.

Example 2.3.16. The following expressions are all equivalent: $2 + 4$, $\frac{12}{2}$, 3×2, $11 - 5$.

2.4. **What are the vocabulary words (i.e. terms) they need to learn?**

- Base
- Exponent
- Exponentiation
- Square (of a number)
- Cube (of a number)
- Superscript (optional)

Let's define these terms:

Definition 2.4.1. **Exponentiation** is a mathematical operation, written as b^n, involving the numbers, the **base** b and the **exponent** n. When n is a positive integer, exponentiation corresponds to repeated multiplication of the base, that is

$$b^n = \underbrace{b \times \cdots \times b}_{n}$$

Definition 2.4.2. For a base b, the exponent 2 (or 2^{nd} power) is called the **square** of b or b squared and written b^2.

Definition 2.4.3. For a base b, the exponent 3 (or 3^{rd} power) is called the **cube** of b or b cubed and written b^3.

Definition 2.4.4. A **superscript** is a character that is slightly above the normal line of type.

2.5. **What technical skills are required to perform the calculations?**

- Addition of integers
- Multiplication of rational numbers
- Finding or identifying the reciprocal of a rational number

2.6. **What concepts must students understand deeply to learn and internalize this concept (not just perform calculations)?**

- Commutivitiy (in multiplication with rational numbers)
- Associativity (in multiplication with rational numbers)
- Multiplicative inverses (i.e. that for all rational numbers x, there exists a number, denoted $\frac{1}{x}$, such that $x \cdot \frac{1}{x} = 1$)
- 0 (the additive identity)
- 1 (the multiplicative identity)

2.7. **What important properties, rules, or identities do students need to learn about the concept?** For any nonzero real numbers a and b and integers n and m:

(1) $a^n a^m = a^{n+m}$
(2) $(a^n)^m = a^{nm}$
(3) $a^n b^n = (ab)^n$
(4) $a^0 = 1$
(5) $a^{-1} = \frac{1}{a^n}$

Here are some tips for how to motivate these properties:

Example 2.7.1. $2^3 \cdot 2^5 = (2 \cdot 2 \cdot 2) \times (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^8$

Example 2.7.2. $(5^3)^4 = 5^3 \times 5^3 \times 5^3 \times 5^3 = (5 \cdot 5 \cdot 5) \times (5 \cdot 5 \cdot 5) \times (5 \cdot 5 \cdot 5) \times (5 \cdot 5 \cdot 5) = 5^{12}$

Example 2.7.3. $3^4 7^4 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 7 \cdot 7 \cdot 7 \cdot 7 = (3 \cdot 7) \times (3 \cdot 7) \times (3 \cdot 7) \times (3 \cdot 7) = (3 \cdot 7)^4$

Remark 2.7.4. Since the standard does not say the base must be an integer students should also work examples with the base being a fraction, base being a negative number, and the exponent a positive integer.

Remark 2.7.5. What does a^0 mean? Let's work an example to figure it out. $3^5 = 3^{5+0} = 3^5 \cdot 3^0$ from Property 1. We know that the only number we can multiply by and get the same number we started with is 1. That is, 1 is the only number we can multiply 3^5 by and the answer is 3^5. So $3^0 = 1$. Is this true if 3 is another number? So we can reason that $a^0 = 1$ for any nonzero number a.

Remark 2.7.6. Now let's figure out what a^{-n} means by looking at 3^{-4}. We know $3^{-4} \cdot 3^4 = 3^{-4+4} = 3^0 = 1$. We also know that if the product of two numbers equals 1 then the numbers are reciprocals of each other. So $3-4$ is the reciprocal of 3^4 and we know we can write the reciprocal of 3^4 as $\frac{1}{3^4}$. Using this reasoning we know that $a^{-n} = \frac{1}{a^n}$.

Example 2.7.7. Here is a list of different types of advanced examples for this standard:

- $\frac{4^3}{5^2} = \frac{64}{25}$
- $\frac{5^3}{5^4} = 5^{3-4} = 5^{-1} = \frac{1}{5}$
- $(-3)^4 = 81$
- $(-3)^{-4} = \frac{1}{81}$
- $(-2)^3 = -8$
- $(-2)^{-3} - \frac{1}{8}$
- $\frac{(-2)^2}{3^{-3}} = (-2)^2 \cdot 3^3 = 4 \cdot 27 = 108$

2.8. **What is the next step (how will this concept be expanded on in future lessons and grades)?**

- This can lead to square root (and further to n^{th} roots).
- The exponents can be rational numbers.
- This can lead to the introduction of irrational numbers (i.e. $2^{\frac{1}{2}} = \sqrt{2}$).

2.9. **What are common misconceptions?**

- Students may confuse the operations for the properties of integer exponents. There is a tendency to memorize rules rather than internalize the concepts behind the laws of exponents.
- Student may incorrectly assume that the value of a number is negative when its exponent is negative.

3. 7^{th} GRADE STANDARD

7.EE.B.3 Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. For example: If a woman making 25 an hour gets a 10% raise, she will make an additional $\frac{1}{10}$ of her salary an hour, or \$2.50, for a new salary of \$27.50. If you want to place a towel bar $9\frac{3}{4}$ inches long in the center of a door that is $27\frac{1}{2}$ inches wide, you will need to place the bar about 9 inches from each edge; this estimate can be used as a check on the exact computation.

4. 6^{th} GRADE STANDARD

6.RPA.1 Understand the concept of a unit rate $\frac{a}{b}$ associated with a ratio $a : b$ with $b \neq 0$, and use rate language in the context of a ratio relationship. For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is $\frac{3}{4}$ cup of flour for each cup of sugar." "We paid \$ 75 for 15 hamburgers, which is a rate of \$ 5 per hamburger." Expectations for unit rates in this grade are limited to non-complex fractions.